# MATH 115 Linear Algebra
# Worksheet 6

December 20, 2020

**Definition:** A set of vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ in $V$ is called a *basis* for $V$ if the following two conditions are satisfied:

- The vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ are linearly independent.
- The vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ span $V$ (that is every element of $V$ is a linear combination of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$).

*Remark:* Note that a basis is a maximal linearly independent set. A basis can also be described as a minimal spanning set.

*Fact/Definition:* Any two bases of a vector space $V$ have the same number of elements and this number is called the *dimension* of $V$.

1. Let $B_1 = \{(1,0),(0,1)\}$, $B_2 = \{(1,1),(-1,1)\}$.
   (a) Show that $B_1$ forms a basis for $\mathbb{R}^2$ and find the components of $(5,-1)$ with respect to the basis $B_1$.
   (b) Show that $B_2$ forms a basis for $\mathbb{R}^2$ and find the components of $(5,-1)$ with respect to the basis $B_2$.

2. Determine whether the following vectors form a basis for $\mathbb{R}^4$.
$$(1,1,0,2),(2,1,3,-1),(-1,1,1,-2),(2,-1,1,2).$$

3. Determine a basis for the subspace of $\mathbb{R}^3$ spanned by the vectors $(1,3,3),(1,5,-1),(2,7,4),(1,4,1)$.

4. Let $S$ be the subspace of $\mathbb{R}^3$ that consists of all solutions to the equation $x_1 - 3x_2 + x_3 = 0$. Determine a basis for $S$, and find $\dim(S)$.

5. Let $A$ be an $m \times n$ matrix. The solution set to the system $A\vec{x} = \vec{0}$ is called the null space of $A$. Find a basis for the null spaces of the following matrices.
$$\text{(a)} \begin{bmatrix} 1 & -1 & 4 \\ 2 & 3 & -2 \\ 1 & 2 & -2 \end{bmatrix} \text{(b)} \begin{bmatrix} 1 & 3 & -2 & 1 \\ 3 & 10 & -4 & 6 \\ 2 & 5 & -6 & -1 \end{bmatrix}.$$

6. Let $p_1(x) = x - 4$ and $p_2(x) = x^2 - x + 3$ be in $P_3$. Determine whether $2x^2 - x + 2$ lies in the space spanned by $p_1(x), p_2(x)$.

**Answers**

1. (a) $(5,-1) = 5(1,0) - 1(0,1)$, (b) $(5,-1) = 2(1,1) - 3(-1,1)$.
2. Yes.
3. $\{(1,3,3),(1,5,-1)\}$.
4. A basis is: $B = \{(3,1,0),(-1,0,1)\}, dim(S) = 2$.
5. (a) $B = \{(-2,2,1)\}$, (b) $B = \{(8,-2,1,0),(8,-3,0,1)\}$.
6. Yes.