MATH 115 Linear Algebra
Worksheet 5

December 20, 2020

1. Are the following sets vectors spaces over $\mathbb{R}$ (with their usual addition and scalar multiplication)?

   (a) Functions $f : \mathbb{R} \to \mathbb{R}$ which take positive values.

   (b) Functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = 0$ for all $x \geq 0$.

   (c) 2×2 real matrices $A = [a_{ij}]$ such that $a_{21} = 0$.

   (d) Upper triangular $n \times n$ real matrices (where $n \geq 1$).

2. For the given vector space V, determine whether the given subset W is a subspace or not.

   (a) $V = M_{n\times n}(\mathbb{R})$, $W =$ lower triangular matrices.

   (b) $V = M_{n\times n}(\mathbb{R})$, $W =$ diagonal matrices.

   (c) $V = M_{n\times n}(\mathbb{R})$, $W =$ invertible matrices.

   (d) $V = M_{n\times n}(\mathbb{R})$, $W =$ non-invertible matrices (for this one, assume $n \geq 2$).

   (e) $V =$ all polynomials on $\mathbb{R}$ with real coefficients, $W =$ polynomials of degree n, where $n \geq 1$ is a fixed number.

   (f) $V = \mathbb{R}^3$, $W =$ all vectors of the form $\vec{v} = (v_1, 0, v_3)$ for some $v_1, v_3 \in \mathbb{R}$ (that is, all vectors whose middle components are 0).

   (g) Same as above, but the middle component is 1.

   (h) $V = \mathbb{R}^n$, $W =$ all vectors $\vec{x} = (x_1, \ldots, x_n)$ such that
   $$x_1 + 2x_2 + 3x_3 + \cdots + nx_n = 0.$$

   (i) V any vector space, $\vec{v}_1, \ldots, \vec{v}_k \in V$ are some fixed elements, $W =$ all vectors $\vec{v}$ of the form
   $$\vec{v} = c_1\vec{v}_1 + \cdots + c_k\vec{v}_k$$
   for some scalars $c_1, \ldots, c_k$.

   (j) $V = P_3 =$ Real-valued polynomials of degree ≤ 3, $W =$ all polynomials of the form $p(x) = ax^2 + 1$ for some $a \in \mathbb{R}$.

3. Let S be the subspace of $\mathbb{R}^3$ consisting of all solutions to the linear equation $x - 2y - z = 0$. Determine a set of vectors that spans S.

4. Determine whether the vector $\vec{v} = (5, 3, -6)$ lies in the subspace of $\mathbb{R}^3$ spanned by the vectors $\vec{v}_1 = (-1, 1, 2), \vec{v}_2 = (3, 1, -4)$.

5. Determine whether the following vector are linearly independent or not.

   (a) $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix}$.

   (b) $\vec{v}_1 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 7 \\ -4 \\ 1 \end{bmatrix}$.

   (c) $\vec{v}_1 = \begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 17 \\ -4 \\ 11 \end{bmatrix}$.

   (d) $\vec{v}_1 = \begin{bmatrix} 1 \\ 10 \\ 14 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 5 \\ 13 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} -2 \\ 5 \\ 1 \end{bmatrix}$.

   (e) $\vec{v}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 2 \\ -3 \\ 0 \\ 0 \end{bmatrix}, \vec{v}_4 = \begin{bmatrix} -6 \\ 9 \\ 0 \\ 5 \end{bmatrix}$.

6. Determine all values of k for which the following vectors are linearly independent in $\mathbb{R}^4$:
   $$(1, 0, 1, k),\ (-1, 0, k, 1),\ (2, 0, 1, 3).$$

7. Let $A = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 5 \\ -1 & -3 & 3 & 0 \end{bmatrix}$.

   (a) Show that the row vectors of A are linearly dependent by finding a dependency relation.

   (b) Show that the column vectors of A are linearly dependent by finding a dependency relation.

8. Let $A = \begin{bmatrix} 3 & 4 & 2 \\ 0 & 1 & 5 \\ 0 & 0 & 2 \end{bmatrix}$.

   (a) Show that the row vectors of A are linearly independent.

   (b) Show that the column vectors of A are linearly independent.

Answers

1. (a) No (b) Yes (c) Yes (d) Yes.

2.

(a) Yes.
(b) Yes.
(c) No.
(d) No.
(e) No.
(f) Yes.
(g) No.
(h) Yes.
(i) Yes.
(j) No.

3. A spanning set is $\{(2,1,0),(1,0,1)\}$.

4. Yes.

5. (a) LI, (b) LD ,(c) LD, (d) LI, (e) LD.

6. $k \neq -1, 2$.

7. (a) $R_3 = 2R_2 - 5R_1$, (b) $3C_1 = C_2$.