MATH 114/116 Linear Algebra
Worksheet 3

March 6, 2022

1. Find the inverses of the following matrices by directly solving the matrix equations, or show that it doesn't exist.

   (a) $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}$ (c) $\begin{bmatrix} -3 & 2 \\ 9 & -6 \end{bmatrix}$.

2. Use Gauss-Jordan Method to compute the inverse of the following matrices:

$$\begin{pmatrix} 2 & -1 \\ 1 & 4 \end{pmatrix}, \begin{pmatrix} 3 & 1 & 1 \\ -2 & 0 & 3 \\ 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 3 & 5 & 1 \\ 1 & 2 & 1 \\ 2 & 6 & 7 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix}.$$

3. If A and B are invertible matrices of the same size then show that

   - $(A^{-1})^{-1} = A$.
   - $(AB)^{-1} = B^{-1}A^{-1}$.
   - $(A^T)^{-1} = (A^{-1})^T$.

4. Given

$$A = \begin{pmatrix} 2 & -1 & 3 \\ 5 & 2 & 1 \\ 3 & -3 & 7 \end{pmatrix}, \; B = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix},$$

   calculate the determinant of the matrices A, B and B^2.

5. Compute the determinants of the following matrices:

   (a) $\begin{pmatrix} 2 & 3 \\ -2 & 6 \end{pmatrix}$ (b) $\begin{pmatrix} -4 & 10 \\ 7 & 3 \end{pmatrix}$

   (c) $\begin{pmatrix} -1 & 2 & -4 \\ -1 & -1 & 5 \\ 2 & 7 & -3 \end{pmatrix}$ (d) $\begin{pmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 8 \end{pmatrix}$

   (e) $\begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{pmatrix}$ (f) $\begin{pmatrix} 1 & -2 & -1 \\ -1 & 5 & 6 \\ 5 & 4 & 5 \end{pmatrix}$

   (g) $\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$ (h) $\begin{pmatrix} 3 & 0 & -3 \\ 2 & 0 & 4 \\ -4 & 0 & 7 \end{pmatrix}$

   (i) $\begin{pmatrix} 2 & 0 & 4 \\ 2 & -1 & 3 \\ 4 & 0 & 8 \end{pmatrix}$ (j) $\begin{pmatrix} -1 & 5 & 14 \\ 0 & 3 & 5 \\ 0 & 0 & -4 \end{pmatrix}$

   (k) $\begin{pmatrix} 16 & -2 & -1 \\ 2 & 5 & 6 \\ 4 & 4 & 5 \end{pmatrix}$ (l) $\begin{pmatrix} 1 & -1 & -3 & 0 \\ 0 & 1 & 5 & 4 \\ 3 & -1 & -2 & 3 \\ -1 & 2 & 8 & 5 \end{pmatrix}$

   (m) $\begin{pmatrix} a & b & a \\ b & a & b \\ a & b & a \end{pmatrix}$ (n) $\begin{pmatrix} 1 & 3 & -1 & 0 & 2 \\ 0 & 2 & -4 & -1 & -6 \\ -2 & -6 & 2 & 3 & 9 \\ 3 & 7 & -3 & 8 & -7 \\ 3 & 5 & 5 & 2 & 7 \end{pmatrix}$

   (m) $\begin{pmatrix} 1 & 3 & 0 & 2 \\ -2 & -5 & 7 & 4 \\ 3 & 5 & 2 & 1 \\ 1 & -1 & 2 & -3 \end{pmatrix}$ (n) $\begin{pmatrix} 1 & 3 & 3 & -4 \\ 0 & 1 & 2 & -5 \\ 2 & 5 & 4 & -3 \\ -3 & -7 & -5 & 2 \end{pmatrix}$

   (o) $\begin{pmatrix} -1 & 0 & 0 & 0 & 2 \\ 0 & 4 & 3 & 2 & -2 \\ 0 & 1 & 2 & 3 & -6 \\ 2 & 2 & 5 & 5 & 1 \\ -4 & -4 & 7 & 9 & 0 \end{pmatrix}$ (p) $\begin{pmatrix} -3 & -2 & 1 & -4 \\ 1 & 3 & 0 & -3 \\ -3 & 4 & -2 & 8 \\ 3 & -4 & 0 & 4 \end{pmatrix}$

6. Suppose that $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & k \end{vmatrix} = 5$. Find the following determinants:

(a) $\begin{vmatrix} 3a & 3b & 3c \\ -5d & -5e & -5f \\ g & h & k \end{vmatrix}$ (b) $\begin{vmatrix} a & b & c \\ d & e & f \\ -2g & -2h & -2k \end{vmatrix}$

(c) $\begin{vmatrix} d & e & f \\ a & b & c \\ g & h & k \end{vmatrix}$ (d) $\begin{vmatrix} a+d & b+e & c+f \\ a & b & c \\ g & h & k \end{vmatrix}$

(e) $\begin{vmatrix} d & e & f \\ a & b & c \\ 3a-g & 3b-h & 3c-k \end{vmatrix}$

7. Verify that $\det(AB) = \det(A)\det(B)$ for the matrices
$A = \begin{pmatrix} -6 & 8 \\ 0 & -2 \end{pmatrix}$, $\quad$ B $= \begin{pmatrix} 6 & -5 \\ -3 & -1 \end{pmatrix}$.

8. Let A and B be two invertible matrices with $\det(A) = \alpha$ and $\det(B) = \beta$. Find the following determinants in terms of α and β using the fact that $\det(AB) = \det(A)\det(B)$.

(a) $\det(2A)$.
(b) $\det(BA)$.
(c) $\det(A^{-1})$.
(d) $\det(B^{-1}AB)$.

9. Give an example to show that it is ***not*** true that $\det(A + B) = \det(A) + \det(B)$.

10. In each part below, find all values of t for which the given matrix is invertible.

(a) $\begin{pmatrix} 1 & t & t^2 \\ 0 & 1 & 2t \\ t & 0 & 2 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos t & -\sin t \\ 0 & \sin t & \cos t \end{pmatrix}$

(c) $\begin{pmatrix} t-1 & -t^2 & \ln t \\ 0 & 1 & 2t \\ 0 & 0 & t-2 \end{pmatrix}$

11. Find the inverses of the matrices

$$A = \begin{bmatrix} 8 & 6 \\ 5 & 4 \end{bmatrix}, \ B = \begin{bmatrix} 7 & 3 \\ -6 & -3 \end{bmatrix}.$$

and use these inverses to determine the solutions of the following systems of linear equations.

(a)

$$\begin{aligned} 8x + 6y &= 2, \\ 5x + 4y &= -1. \end{aligned}$$

(b)

$$\begin{aligned} 7x + 3y &= -9, \\ -6x - 3y &= 4. \end{aligned}$$

12. Suppose you solve $A\vec{x} = \vec{b}$ for three different values of $\vec{b}$:
$A\vec{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $A\vec{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $A\vec{x}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. If the three solutions $\vec{x}_1, \vec{x}_2, \vec{x}_3$ are columns of a matrix X what is AX?

Answers

1. (a) $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$.

(b) $\frac{1}{10}\begin{bmatrix} 4 & -2 \\ 3 & 1 \end{bmatrix}$.

(c) One can directly show that there is no matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ whose product with the given matrix is I_2. If we had

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} -3 & 2 \\ 9 & -6 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

this would mean $-3a + 9b = 1$ and $2a - 3b = 0$. This has no solution (check and see!).

2. $\begin{bmatrix} 4/9 & 1/9 \\ -1/9 & 2/9 \end{bmatrix}$, $\begin{bmatrix} 3/8 & -1/8 & -3/8 \\ -3/8 & 1/8 & 11/8 \\ 1/4 & 1/4 & -1/4 \end{bmatrix}$, $\begin{bmatrix} 1 & 1 & -1 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & -1 & 1 & 0 \end{bmatrix}$, $\begin{bmatrix} 8 & -29 & 3 \\ -5 & 19 & -2 \\ 2 & -8 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & -3 & 2 \\ -3 & 3 & -1 \\ 2 & -1 & 0 \end{bmatrix}$.

3.

4. $\det(A) = 3$, $\det(B) = -4$, $\det(B^2) = 16$.

5.

6. (a)-75, (b) -10, (c) -5, (d) -5, (e) 5.

7.

8. (a) $2^n\alpha$, (b) $\alpha\beta$, (c) $\frac{1}{\alpha}$, (d) α.

9. For example take $A = B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

10. (a) $\mathbb{R} - \{\sqrt[3]{-2}\}$, (b) $\mathbb{R}$, (c) $\mathbb{R} - \{1, 2\}$.

11. (a) $A^{-1} = \begin{bmatrix} 2 & -3 \\ -\frac{5}{2} & 4 \end{bmatrix}$ and the solution is $\begin{bmatrix} 7 \\ -9 \end{bmatrix}$.

    (b) $B^{-1} = \begin{bmatrix} 1 & 1 \\ -2 & -\frac{7}{3} \end{bmatrix}$ and the solution is $\begin{bmatrix} -5 \\ \frac{26}{3} \end{bmatrix}$.

12. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.