MATH 115 Linear Algebra
Worksheet 2

November 5, 2020

1. For the matrix A and the vector (column-matrix) v given below, compute the product Av.

   (a) $A = \begin{pmatrix} 2 & 5 \\ -3 & -1 \end{pmatrix}$ and $v = \begin{pmatrix} 5 \\ 6 \end{pmatrix}$.

   (b) $A = \begin{pmatrix} 1 & -3 \\ -2 & 1 \\ 3 & -1 \end{pmatrix}$ and $v = \begin{pmatrix} 5 \\ -5 \end{pmatrix}$.

   (c) $A = \begin{pmatrix} 6 & 2 \\ -3 & -1 \end{pmatrix}$ and $v = \begin{pmatrix} 7 \\ -1 \end{pmatrix}$.

   (d) $A = \begin{pmatrix} 5 & -3 & 1 \\ -2 & 1 & 4 \\ 1 & 0 & -2 \end{pmatrix}$ and $v = \begin{pmatrix} 0 \\ 22 \\ -11 \end{pmatrix}$.

2. Consider the square matrices $A = \begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} -1 & 3 \\ 0 & 4 \end{pmatrix}$.

   (a) Is it true that $AB = BA$?

   (b) Compute A^2 and B^2.

3. Let A, B be square matrices of the same size. Then $(A+B)^2$ may not be equal to the matrix $A^2 + 2AB + B^2$. Observe this for $A = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 \\ 3 & 0 \end{pmatrix}$.

4. For the matrices

$$A = \begin{pmatrix} 3 & -2 & 1 \\ -1 & 0 & 5 \end{pmatrix}, \ B = \begin{pmatrix} 1 & 5 & 1 \\ 2 & 3 & -1 \end{pmatrix} \text{ and,}$$

$$C = \begin{pmatrix} 2 & -2 & 3 \\ -1 & 1 & 1 \\ 3 & 1 & 1 \end{pmatrix}$$

   answer the following questions:

   (a) Find, if possible,

$$A+B,\ B+C,\ 5A,\ AB,\ BC,\ CB,\ A^2,\ C^2.$$

   (b) Find a matrix D such that

$$A - 3B + D = BC^2 + 2A.$$

   (c) Find B^T, C^T, $(BC)^T$.

5. Let $A = \begin{pmatrix} 3 & -1 \\ -5 & a \end{pmatrix}$. If $A^2 - 2A - 8I_2 = 0_2$ determine a.

6. If $v = \begin{pmatrix} 2 \\ 3i \\ 1-i \end{pmatrix}$ and $w = \begin{pmatrix} -1+i \\ 2 \\ 3-i \end{pmatrix}$ find $v^T w$ and $w^T w$.

7. If $A = \begin{pmatrix} x & 1 \\ -2 & y \end{pmatrix}$, determine all values of x and y for which $A^2 = A$.

8. Let $A = \begin{pmatrix} 2 & 2 & 1 \\ 2 & 5 & 2 \\ 1 & 2 & 2 \end{pmatrix}$, $B = \begin{pmatrix} -x & -y & z \\ 0 & y & 2z \\ x & -y & z \end{pmatrix}$.

   Find all values x, y, z such that

$$B^T AB = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 7 \end{pmatrix}.$$

9. The *commutator* of two square matrices A and B is defined as $[A, B] = AB - BA$. Compute $[A, B]$ for the following matrices:

   (a) $A = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 5 \\ -7 & 4 \end{pmatrix}$.

   (b) $A = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$, $B = \frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$.

10. Find the intersection point of the lines below (if there is any).

    (a) $L_1 : 2x - 3y + 12 = 0,\ \ L_2 : 3x + 5y - 8 = 0.$

    (b) $L_1 : y = -3x + 2,\ \ L_2 : y = 9x + 4.$

    (c) $L_1 : 2x - 5y - 6 = 0,\ \ L_2 : -6x + 15y - 6 = 0.$

11. Show that the following system of equations do not have any solution. What is the geometric explanation of this?

    (a)
$$\begin{aligned} -3x + 6y &= 11, \\ 6x - 12y &= 25. \end{aligned}$$

    (b)
$$\begin{aligned} x - y &= 1, \\ y - x &= 7. \end{aligned}$$

12. Find the value of k for which the following system of equations have an infinite number of solutions.
$$\begin{aligned} 5y + 3x &= 27, \\ 12x + 20y &= 3k. \end{aligned}$$

Answers

1. (a) $Av = 5\begin{bmatrix} 2 \\ -3 \end{bmatrix} + 6\begin{bmatrix} 5 \\ -1 \end{bmatrix} = \begin{bmatrix} 40 \\ -21 \end{bmatrix}.$

    (b) $Av = 5\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} - 5\begin{bmatrix} -3 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 20 \\ -15 \\ 20 \end{bmatrix}.$

    (c) $Av = 7\begin{bmatrix} 6 \\ -3 \end{bmatrix} - 1\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 40 \\ -20 \end{bmatrix}.$

    (d) $Av = 0\begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix} + 22\begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} - 11\begin{bmatrix} 1 \\ 4 \\ -2 \end{bmatrix} = \begin{bmatrix} -77 \\ -22 \\ 22 \end{bmatrix}.$

2. (a) $\begin{bmatrix} -5 & 8 \\ -4 & 12 \end{bmatrix}.$

    (b) $A^2 = \begin{bmatrix} 3 & 5 \\ -5 & 8 \end{bmatrix},\ B^2 = \begin{bmatrix} 1 & 9 \\ 0 & 16 \end{bmatrix}.$

3.

4. $A + B = \begin{bmatrix} 4 & 3 & 2 \\ 1 & 3 & 4 \end{bmatrix}$, $B + C$ is undefined, $5A = \begin{bmatrix} 15 & -10 & 5 \\ -5 & 0 & 25 \end{bmatrix}$, AB is undefined, $BC = \begin{bmatrix} 0 & 4 & 9 \\ -2 & -2 & 8 \end{bmatrix}$, CB is undefined, A^2 is undefined, $C^2 = \begin{bmatrix} 15 & -3 & 7 \\ 0 & 4 & -1 \\ 8 & -4 & 11 \end{bmatrix}.$

5. $a = -1.$

6. $4i, 12 - 8i.$

7. $x = 2, y = -1$ and $x = -1, y = 2.$

8. $x = \pm\sqrt{2}/2, y = \pm\sqrt{3}/3, z = \pm\sqrt{6}/6.$

9. (a) 0. (b) $\frac{1}{2}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$

10. (a) $(x, y) = -\frac{36}{19}, \frac{52}{19}.$

    (b) $(x, y) = -\frac{1}{6}, \frac{5}{2}.$

    (c) No solution.

11. You'll see that there are no solutions in (a) and (b). That means they are parallel to each other.

12. $k = 36.$