The Inverse of A Matrix

Recall from Algebra how to solve scaler linear equation $ax = b$ for x:

Case I Suppose $a \neq 0 \Rightarrow$

$$ax = b$$
$$a^{-1} a x = a^{-1} b$$
$$(a^{-1} a) x = a^{-1} b$$
$$1 x = a^{-1} b$$
$$x = a^{-1} b = \frac{b}{a}$$

Case II Suppose $a = 0 \Rightarrow$

$ax = b \Rightarrow (0)x = b \Rightarrow 0 = b$ → if $b = 0 \Rightarrow$ infinitely many sol'n

→ if $b \neq 0 \Rightarrow$ no solution

Question Is there an inverse of a matrix A when solving linear sys. $A\vec{x} = \vec{b}$?

Answer Provided the linear system /matrix is square, then maybe:

Definition (Inverse of a Square Matrix)

Let A be an $n \times n$ square matrix & I be the $n \times n$ identity matrix.

Then A is invertible, if there exists an $n \times n$ matrix A^{-1} such that

$$A^{-1}A = AA^{-1} = I.$$

A^{-1} is called the inverse of A.

If A does not have an inverse, A is called singular (AKA noninvertible)

Non-square matrices do not have inverses.

Question Is it possible for an invertible matrix to have two or more inverses?

Answer No!! There will be one and only one inverse:

Theorem If $n \times n$ square matrix A is invertible, then its inverse A^{-1} is unique.

How to Systematically find the Inverse of a Matrix?

Consider finding inverse of $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$,

Then, if the inverse $A^{-1} = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}$ exists:

$$AA^{-1} = I \Rightarrow \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{cases} x_{11} + 2x_{21} = 1 \\ 3x_{11} + 4x_{21} = 0 \end{cases} \text{ and } \begin{cases} x_{12} + 2x_{22} = 0 \\ 3x_{12} + 4x_{22} = 1 \end{cases}$$

$\Rightarrow$ Perform Gauss-Jordan on

$$\left[\begin{array}{cc|c} 1 & 2 & 1 \\ 3 & 4 & 0 \end{array}\right] \ \& \ \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 3 & 4 & 1 \end{array}\right]$$

$\Rightarrow$ Perform Gauss-Jordan on $\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right]$

$= [A \mid I]$

If A^{-1} exists, then linear systems have unique solutions $\Rightarrow$ RREF$(A) = I$

If A is singular, then linear sysms have no solution $\Rightarrow$ RREF$(A) \neq I$

Question: So how to find the inverse of A (if it exists)?

Answer: Apply Gauss-Jordan as follows:

GIVEN: Square $n \times n$ matrix A

TASK: Find A^{-1} if it exists, otherwise conclude A is singular.

(1) Form augmented matrix $[A \mid I]$ where I is $n \times n$ identity matrix.

(2) Apply Gauss-Jordan Elimination to $[A \mid I]$:

If $RREF(A) \neq I$ then A is singular.

If $RREF(A) = I$ then $[A \mid I] \xrightarrow{G.J.} [I \mid A^{-1}]$

Example ① $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \Rightarrow A^{-1} = ?$

$$[A \mid I] = \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -2 & -3 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & -2 & -3 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & 1 & \frac{3}{2} & \frac{-1}{2} \end{array}\right] = [I \mid A^{-1}]$$

$$\Rightarrow A^{-1} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}$$

(2) Using Gauss Jordan find the inverse of $A=\begin{bmatrix}1&1\\3&3\end{bmatrix}$.

$$[A \mid I] = \left[\begin{array}{cc|cc}1&1&1&0\\3&3&0&1\end{array}\right] \sim \left[\begin{array}{cc|cc}1&1&1&0\\0&0&-3&1\end{array}\right] = [RREF(A) \mid B]$$

Since $RREF(A) \neq I$, A^{-1} doesnot exist

$\Rightarrow A$ singular.

Inverse of a 2X2 Matrix

Let $A = \begin{bmatrix}a&b\\c&d\end{bmatrix}$ be a 2×2 matrix s.t.

$a, b, c, d \in \mathbb{R}$.

Then $\begin{cases} \text{If } ad-bc \neq 0, \text{ then } A^{-1} = \dfrac{1}{ad-bc}\begin{bmatrix}d&-b\\-c&a\end{bmatrix} \\ \text{If } ad-bc=0, \text{ then } A \text{ is singular.} \end{cases}$

Properties of Inverses

Theorem Let A, B be $n \times n$ invertible matrices, k be positive integer, and $\alpha \neq 0$.

Then A^{-1}, A^k, αA, A^T, AB are all invertible and the following are true:

(1) $(A^{-1})^{-1} = A$ (2) $(A^k)^{-1} = (A^{-1})^k$

(3) $(\alpha A)^{-1} = \frac{1}{\alpha} A^{-1}$ (4) $(A^T)^{-1} = (A^{-1})^T$

(5) $(AB)^{-1} = B^{-1}A^{-1}$

* Let $a, b \in \mathbb{R}$ & $c \neq 0$. Then

$$ac = bc \Rightarrow a = b$$
$$ca = cb \Rightarrow a = b$$

Question Is there a similar cancelling behavior for matrix product?

Answer: Yes, provided the matrix to be cancelled is invertible:

Theorem Let C be an invertible matrix and A, B have compatible shapes. Then

* if $AC = BC$, then $A = B$
* if $CA = CB$, then $A = B$

Remember, for $AC = BC$ to imply $A = B$, C must be invertible.

Otherwise, it's possible for $AC = BC$ yet $A \neq B$:

Consider $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$, $B = \begin{bmatrix} 17 & 2 & -1 \\ 20 & 9 & 0 \\ 37 & 5 & 3 \end{bmatrix}$, $C = \begin{bmatrix} 1 & -2 & 2 \\ 2 & -4 & 4 \\ 4 & -8 & 8 \end{bmatrix}$.

Then clearly $A \neq B$ and yet

$$AC = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 1 & -2 & 2 \\ 2 & -4 & 4 \\ 4 & -8 & 8 \end{bmatrix} = \begin{bmatrix} 17 & -34 & 34 \\ 38 & -76 & 76 \\ 59 & -118 & 118 \end{bmatrix}$$

$$B.C = \begin{bmatrix} 17 & 2 & -1 \\ 20 & 9 & 0 \\ 37 & 5 & 3 \end{bmatrix} \begin{bmatrix} 1 & -2 & 2 \\ 2 & -4 & 4 \\ 4 & -8 & 8 \end{bmatrix} = \begin{bmatrix} 17 & -34 & 34 \\ 38 & -76 & 76 \\ 59 & -118 & 118 \end{bmatrix}$$

Solving Square Linear System $A\vec{x} = \vec{b}$ via A^{-1}

How can A^{-1} be used to solve square linear system $A\vec{x} = \vec{b}$?

Theorem Let A be an invertible matrix.

Then square linear system $A\vec{x} = \vec{b}$ has unique solution given by $\vec{x} = A^{-1}\vec{b}$.

Remark This is useful when solving several square linear systems with the same matrix A & different RHS $\vec{b}$'s since A^{-1} only has to be found once:

$$A\vec{x} = \vec{b}_1 \longrightarrow \vec{x} = A^{-1}\vec{b}_1$$
$$A\vec{x} = \vec{b}_2 \longrightarrow \vec{x} = A^{-1}\vec{b}_2$$
$$\vdots \qquad\qquad \vdots$$

If A is invertible & B, X have compatible shapes s.t. product AX is well-defined then A^{-1} can be used to solve square matrix equation

$$AX = B \text{ for } X \rightsquigarrow X = A^{-1}B$$

Similarly $XA = B \Rightarrow X = BA^{-1}$

Example (1) Find the inverse of the matrix.

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ -6 & 2 & 3 \end{bmatrix}$$

$$[A \mid I] = \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ -6 & 2 & 3 & 0 & 0 & 1 \end{array}\right] \quad (-1)R_1 + R_2 \to R_2 \quad \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ -6 & 2 & 3 & 0 & 0 & 1 \end{array}\right]$$

$$(6)R_1 + R_3 \to R_3 \quad \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & -4 & 3 & 6 & 0 & 1 \end{array}\right] \quad (4)R_2 + R_3 \to R_3 \quad \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & -1 & 2 & 4 & 1 \end{array}\right]$$

$$(-1)R_3 \to R_3 \quad \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -2 & -4 & -1 \end{array}\right] \quad R_3 + R_2 \to R_2 \quad \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -3 & -3 & -1 \\ 0 & 0 & 1 & -2 & -4 & -1 \end{array}\right]$$

$$R_2 + R_1 \to R_1 \quad \left[\underbrace{\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}}_{I} \middle| \underbrace{\begin{array}{ccc} -2 & -3 & -1 \\ -3 & -3 & -1 \\ -2 & -4 & -1 \end{array}}_{A^{-1}}\right]$$

A is invertible & $A^{-1} = \begin{bmatrix} -2 & -3 & -1 \\ -3 & -3 & -1 \\ -2 & -4 & -1 \end{bmatrix}$

(2) Find the inverse of the matrix.

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ -2 & 3 & -2 \end{bmatrix}$$

$$[A \mid I] = \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 3 & -1 & 2 & 0 & 1 & 0 \\ -2 & 3 & -2 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & -7 & 2 & -3 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 1 \end{array}\right]$$

$$\underbrace{\begin{matrix} 1 & 2 & 0 \\ 0 & -7 & 2 \\ 0 & 0 & 0 \end{matrix}}_{\neq I}$$

$\therefore A$ is singular
(noninvertible)

(3) let $A = \begin{bmatrix} 1 & 1 \\ 2 & 4 \end{bmatrix}$ $\quad A^{-2} = ?$

$$A^{-2} = (A^{-1})^2 = (A^2)^{-1}$$

$$A^2 = \begin{bmatrix} 3 & 5 \\ 10 & 18 \end{bmatrix} \Rightarrow (A^2)^{-1} = \frac{1}{4}\begin{bmatrix} 18 & -5 \\ -10 & 3 \end{bmatrix} = \begin{bmatrix} 9/2 & -5/4 \\ -5/2 & 3/4 \end{bmatrix}$$

OR

$$A^{-1} = \frac{1}{2}\begin{bmatrix} 4 & -1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1/2 \\ -1 & 1/2 \end{bmatrix} \Rightarrow (A^{-1})^2 = \begin{bmatrix} 9/2 & -5/4 \\ -5/2 & 3/4 \end{bmatrix}$$