Equations of Lines & Circles

DISTANCE BETWEEN TWO POINTS (x_0, y_0) AND (x_1, y_1) : $\sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2}$

EQUATION OF A (GENERAL) LINE:

- Assume line contains points (x_0, y_0), (x_1, y_1), **y-intercept** $(0, b)$, and **x-intercept** $(a, 0)$
- Then the **slope** is $m = \dfrac{y_1 - y_0}{x_1 - x_0}$
- Point-slope form : $y - y_0 = m(x - x_0)$
- Slope-intercept form : $y = mx + b$

EQUATION OF A HORIZONTAL LINE: $y = k$ (The slope of a horizontal line is zero)

EQUATION OF A VERTICAL LINE: $x = h$ (The slope of a vertical line DNE [**D**oes **N**ot **E**xist])

PARALLEL LINES:

- Two **vertical** lines are **parallel**.
- Two **non-vertical** lines ℓ_1, ℓ_2 are **parallel** $\iff \ell_1 \,||\, \ell_2 \iff$ their slopes $m_1 = m_2$

PERPENDICULAR LINES: **slanted** line $\iff$ line that's neither horizontal nor vertical

- **Horizontal** lines are **perpendicular** to **vertical** lines.
- Two **slanted** lines ℓ_1, ℓ_2 are **perpendicular** $\iff \ell_1 \perp \ell_2 \iff$ their slopes $m_1 m_2 = -1$

INTERSECTION OF TWO LINES:

- Finding **intersection** of 2 lines $\ell_1, \ell_2 \iff$ solving a **system** of 2 linear equations
- 3 cases :
  (i) $\ell_1 \,||\, \ell_2 \iff$ no solution (ii) ℓ_1, ℓ_2 coincident $\iff$ infinitely many solutions (iii) ℓ_1, ℓ_2 intersect $\iff$ one solution
- Solve system of 2 linear equations by using **substitution** : Solve one equation for either variable, then plug expression into same variable of other equation, then solve for other variable.

EXAMPLE: Suppose a line contains points $(1, 8)$ and $(4, -2)$. Determine its equation in slope-intercept form.

Let points $(x_0, y_0) = (1, 8)$ and $(x_1, y_1) = (4, -2)$. Then, slope $m = \dfrac{y_1 - y_0}{x_1 - x_0} = \dfrac{(-2) - (8)}{(4) - (1)} = -\dfrac{10}{3}$

Since we do not immediately recognize the y-intercept of this line, plug the values x_0, y_0, m into the point-slope form :

$$y - y_0 = m(x - x_0) \implies y - 8 = -\frac{10}{3}(x - 1) \implies y = 8 - \frac{10}{3}x + \frac{10}{3} \implies \boxed{y = -\frac{10}{3}x + \frac{34}{3}}$$

EXAMPLE: Suppose a line ℓ has x-intercept -10 and y-intercept 5. Determine its equation in slope-intercept form.

First, "line ℓ has x-intercept -10" $\iff$ "line ℓ contains point $(-10, 0)$" $\iff$ point $(-10, 0) \in \ell$

Next, "line ℓ has y-intercept 5" $\iff$ "line ℓ contains point $(0, 5)$" $\iff$ point $(0, 5) \in \ell$

Thus, let points $(x_0, y_0) = (-10, 0)$ and $(x_1, y_1) = (0, 5)$. Then, slope $m = \dfrac{y_1 - y_0}{x_1 - x_0} = \dfrac{(5) - (0)}{(0) - (-10)} = \dfrac{1}{2}$

Since we were given the y-intercept of line ℓ $(b = 5)$, plug the values m, b into the slope-intercept form :

$$\ell : y = mx + b \implies \boxed{\ell : y = \frac{1}{2}x + 5}$$

EXAMPLE: Let line ℓ_1 contain point $(-3, -12)$ and be perpendicular to $\ell_2 : -2x + y = 4$. Find slope-intercept form of ℓ_1.

First, $\ell_1 \perp \ell_2 \iff$ slopes $m_1 m_2 = -1$. Now, find slope m_2 of line ℓ_2: $-2x + y = 4 \implies y = 2x + 4 \implies m_2 = 2$

Hence, $m_1 m_2 = -1 \implies m_1(2) = -1 \implies m_1 = -\dfrac{1}{2} \implies$ slope of line ℓ_1 is $m_1 = -\dfrac{1}{2}$

Let $(x_0, y_0) = (-3, -12) \in \ell_1$. Then, plug the values x_0, y_0, m_1 into the point-slope form :

$$\ell_1 : y - y_0 = m_1(x - x_0) \implies \ell_1 : y - (-12) = -\frac{1}{2}(x - (-3)) \implies \ell_1 : y = -12 - \frac{1}{2}x - \frac{3}{2} \implies \boxed{\ell_1 : y = -\frac{1}{2}x - \frac{27}{2}}$$

EQUATION OF A CIRCLE:

- Assume circle has center (h, k) and radius $r > 0$
- Standard form : $(x-h)^2 + (y-k)^2 = r^2$
- Complete the square of $x^2 + bx$ (where $b > 0$) by adding term $\left(\frac{b}{2}\right)^2 \implies x^2 + bx + \left(\frac{b}{2}\right)^2 = \left(x + \frac{b}{2}\right)^2$
- Complete the square of $x^2 - bx$ (where $b > 0$) by adding term $\left(\frac{b}{2}\right)^2 \implies x^2 - bx + \left(\frac{b}{2}\right)^2 = \left(x - \frac{b}{2}\right)^2$

EXAMPLE: Find the center & radius of the circle $x^2 - 2x + y^2 + \sqrt{5}y - 10 = 0$

The main task is to rewrite equation $x^2 - 2x + y^2 + \sqrt{5}y - 10 = 0$ in standard form $(x-h)^2 + (y-k)^2 = r^2$

First, move the constant -10 to the RHS of equation : $x^2 - 2x + y^2 + \sqrt{5}y = 10$

Complete the square in x : $x^2 - 2x + \left(\frac{-2}{2}\right)^2 = x^2 - 2x + 1 = (x-1)^2$

Complete the square in y : $y^2 + \sqrt{5}y + \left(\frac{\sqrt{5}}{2}\right)^2 = y^2 + \sqrt{5}y + \frac{5}{4} = \left(y + \frac{\sqrt{5}}{2}\right)^2$

Hence, we have $x^2 - 2x + [1-1] + y^2 + \sqrt{5}y + \left[\frac{5}{4} - \frac{5}{4}\right] = 10 \implies (x^2 - 2x + 1) + \left(y^2 + \sqrt{5}y + \frac{5}{4}\right) - 1 - \frac{5}{4} = 10$

$\implies (x-1)^2 + \left(y + \frac{\sqrt{5}}{2}\right)^2 = 10 + 1 + \frac{5}{4} \implies (x-1)^2 + \left(y + \frac{\sqrt{5}}{2}\right)^2 = \frac{49}{4} \implies (x-1)^2 + \left(y + \frac{\sqrt{5}}{2}\right)^2 = \left(\frac{7}{2}\right)^2$

$\implies (x-1)^2 + \left[y - \left(-\frac{\sqrt{5}}{2}\right)\right]^2 = \left(\frac{7}{2}\right)^2$, which is in standard form.

Therefore, $\boxed{\text{circle has its center at point } \left(1, -\frac{\sqrt{5}}{2}\right) \text{ and a radius of } \frac{7}{2}}$

EXAMPLE: Find the center & radius of the circle $2x^2 - 12x + 2y^2 + 8y - 24 = 0$

The main task is to rewrite equation $2x^2 - 12x + 2y^2 + 8y - 24 = 0$ in standard form $(x-h)^2 + (y-k)^2 = r^2$

Notice if the LHS of standard form is expanded, we get $(x^2 - 2hx + h^2) + (y^2 - 2ky + k^2) = r^2$

Moreover, observe that the x^2 and y^2 terms of standard form have coefficient one,

so in our given equation, the x^2 and y^2 terms must also have coefficient one.

Hence, divide both sides of equation by 2 : $2x^2 - 12x + 2y^2 + 8y - 24 = 0 \implies x^2 - 6x + y^2 + 4y - 12 = 0$

Now, move the constant -12 to the RHS of equation : $x^2 - 6x + y^2 + 4y = 12$

Complete the square in x : $x^2 - 6x + \left(\frac{-6}{2}\right)^2 = x^2 - 6x + 9 = (x-3)^2$

Complete the square in y : $y^2 + 4y + \left(\frac{4}{2}\right)^2 = y^2 + 4y + 4 = (y+2)^2$

Hence, we have $x^2 - 6x + [9-9] + y^2 + 4y + [4-4] = 12 \implies (x^2 - 6x + 9) + (y^2 + 4y + 4) - 9 - 4 = 12$

$\implies (x-3)^2 + (y+2)^2 = 12 + 9 + 4 \implies (x-3)^2 + (y+2)^2 = 25$

$\implies (x-3)^2 + [y - (-2)]^2 = 5^2$, which is in standard form.

Therefore, $\boxed{\text{circle has its center at point } (3, -2) \text{ and a radius of } 5}$

EXAMPLE: The equation $(x-1)^2 + (y+10)^2 = 0$ does not describe a circle. Why not???

Well, this equation certainly looks like standard form $(x-h)^2 + (y-k)^2 = r^2$ with $h = 1$ and $k = -10$,

but $r = 0$, meaning the radius of this false circle is zero! We know circles must have **positive** radii (that is, $r > 0$).

In fact, the graph of $(x-1)^2 + (y+10)^2 = 0$ is a **single point** at $(h, k) = (1, -10)$!!